Effect of Support Elasticity on the Bending of Axisymmetric Plates

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The problem of flexure of an elastically supported axisymmetric plate carrying lateral load is solved in closed form via energy minimization technique. A second solution obtained directly by integration of the equations of classical plate theory in terms of *ber*, *bei*, *ker*, and *kei* functions is also given. The effect of support width and stiffness is examined by way of a specific problem. The stress couples as well as deflection given by the two solutions are in close agreement for a wide range of support widths and stiffnesses.

Nomenclature

a = radius of the circular plate
M_r = radial bending moment
M_θ = circumferential bending moment
ρ = dimensionless radius, r/a
V = shear stress resultant
q^+ = load intensity on the bottom surface of the plate
q^- = load intensity on the top surface of the plate
D = flexural rigidity of the plate
w = plate deflection
ν = Poisson's ratio for the plate material
E = Young's modulus for the plate material
h = plate thickness
c = radius of the inner edge of support
m = dimensionless ratio, c/a
q_o = uniform intensity of load on the top surface
$\bar{w}$ = dimensionless deflection defined as wD/q_oc^4
$\bar{M}_r$ = dimensionless radial moment defined as M_r/q_oc^2
$\bar{V}$ = dimensionless shear stress resultant defined as V/q_0a
$\bar{A}_1, \ldots \bar{A}_6$ = constants of integration
I = energy integral
$\bar{w}_s$ = value of $\bar{w}$ in the support region
K = foundation modulus of the support
K' = defined by $K(a-c)$
Δ = function of $E, \nu, K, h/c$, and m as defined in the text
p_o = initial uniform clamping pressure in the support
γ = defined in the text as $(2Ka^4/D)^{1/4}$
$\bar{w}_1, \bar{M}_{r1}, \bar{V}_1$ = quantities corresponding to plate region within the support
$\bar{w}_2, \bar{M}_{r2}, \bar{V}_2$ = quantities corresponding to plate in the support region
$(\ )'$ = differentiation with respect to ρ

Introduction

THE problem of a plate supported elastically along the edge is a realistic one. One seldom encounters such ideal support conditions as "simply supported" or "clamped." The practical support conditions fall between these two and it becomes important in many applications to account for the deformation of the support. In particular, the mounting of glass plates presents many engineering problems with respect to defining support characteristics and their effect on the stress distribution in the plate. In a recent paper[1] Gulati and Buehl treated the problem of a rectangular plate supported elastically along its edges and carrying a uniform windload. The solution was obtained via an energy method and was based on the assumption that the plate slope in the supported region remained constant. This latter assumption was justified due to narrowness of the support width and simplified the analysis considerably.

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The purpose of the present paper is to examine the effect of constant slope assumption by considering the flexure of an axisymmetric plate. The choice of such geometry greatly simplifies the Ritz solution and the added assumption of constant slope in the support region leads to a closed form solution. The exact solution of this problem within the scope of classical plate theory is also deduced by dividing the plate into two regions, the region bounded by the inner edge of the support and the region containing the support (supports encountered in practice have a finite width), and satisfying the continuity and boundary conditions. Such plates are often encountered in ships (portholes for example) as well as industrial processing equipment (as sideports). The exact solution involves ber, bei, ker, and kei functions and is similar to that of a plate on an elastic foundation discussed previously by Naghdi and Rowley[2] and Frederick.[3] The problem of bending and buckling of elastically restrained circular and ring-shaped plates has been treated by Reismann.[4,5]

It is shown that the assumption of constant slope in the region of support is reasonable for support widths as large as 0.2 a, where a is the characteristic radius of the plate. Since for all practical cases the support widths <0.2 a, it is concluded that the energy formulation with its simple solution is quite adequate.

Analysis

The present investigation is restricted to thin axisymmetric plates for which the equations of classical plate theory are

$$\begin{aligned} M_r' + (M_r - M_\theta)/\rho &= aV \\ (\rho V)' &= a\rho(q^+ - q^-) \\ M_r &= -(D/a^2)\,[w'' + (\nu/\rho)w'] \\ M_\theta &= -(D/a^2)\left[\left(\frac{w'}{\rho}\right) + \nu w''\right] \\ V &= -(D/a^3)\,[w''' + (w''/\rho) - (w'/\rho^2)] \end{aligned} \tag{1}$$

where M_r and M_θ are the radial and circumferential bending moments, respectively; V is the nonvanishing shear stress resultant; w is the deflection of the middle surface of the plate and is taken positive downward; q^+ and q^- are the load intensities on the bottom and top surfaces of the plate; ρ is the dimensionless radius r/a, a being the characteristic radius of the plate; prime denotes differentiation with respect to ρ and D is the flexural rigidity defined by

$$D = Eh^3/12(1-\nu^2)$$

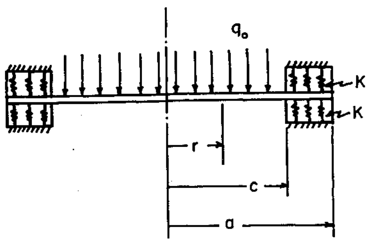

Fig. 1 Axisymmetric plate on elastic support.

in which h is the plate thickness and E and ν are, respectively, the elastic modulus and Poisson's ratio for the plate material.

The solution of Eqs. (1) for a circular plate supported elastically along the edge and carrying a uniform load q_o (Fig. 1) is given by

$$\bar{w} = \rho^4/64m^4 + (\bar{A}_1/4)\rho^2 + \bar{A}_2$$
$$\bar{M}_r = -[(3+\nu)/16m^2]\rho^2 - (m^2/2)(1+\nu)\bar{A}_1 \qquad (2)$$
$$\bar{V} = -\rho/2$$

where

$$\bar{w} = wD/q_oc^4,\ \bar{M}_r = M_r/q_oc^2,\ \bar{V} = V/q_oa$$

and A_1 and A_2 are constants of integration, as yet unknown. The above solution is valid only for the region $0 \le \rho \le m$ where $m = c/a$. For the region $m \le \rho \le 1$, one can solve Eqs. (1) in terms of ber, bei, ker, and kei functions, but the solution, though exact, is rather involved. On the other hand, one may approximate the solution by assuming that the plate slope in this region remains constant at $w'(m)$, due to smallness of the quantity $(1 - m)$ for sufficiently small support width, and obtain expressions for $\bar{A}_1$ and $\bar{A}_2$ by minimizing the energy stored in the bent plate.

Solution by Ritz Method

In terms of dimensionless deflection $\bar{w}$ the strain energy integral for the bent plate is given by

$$I = \frac{\pi q_o^2c^6}{Dm^2}\left(\int_0^m\left[m^4\left\{\left(\bar{w}'' + \frac{\bar{w}'}{\rho}\right)^2 - 2(1-\nu)\frac{\bar{w}'}{\rho}\bar{w}''\right\} - 2\bar{w}\right]\rho d\rho + \int_m^1\left[m^4\left(\frac{\bar{w}_s'}{\rho}\right)^2 + \left(\frac{2Kc^4}{D}\right)\bar{w}_s^2\right]\rho d\rho\right) \qquad (3)$$

where $\bar{w}_s$ denotes the plate deflection in the support region $m \le \rho \le 1$ and is assumed to be of the form

$$\bar{w}_s = \bar{w}(m) + (\rho - m)\bar{w}'(m) \qquad (4)$$

and $2K$ is the foundation modulus of the support (lb/in.^{3}) which is assumed to behave like a linear spring. It should be emphasized that Eq. (4) is valid for sufficiently small support widths. The exact solution will be compared with Eq. (4) and the validity of the assumption leading to Eq. (4) will be critically examined.

Without going into detail, we record the expressions for $\bar{A}_1$ and $\bar{A}_2$ obtained by minimization of integral (3) after substitution of Eqs. (2) and (4);

$$\bar{A}_1 = \frac{1}{\Delta}\left[\frac{3m}{8(1-\nu^2)}\left(\frac{1+m}{1-m}\right)\left(\log m - \nu - \frac{1}{3}\right)\left(\frac{E}{Ka}\right)\times \left(\frac{h}{c}\right)^3 - \frac{2\left(\frac{E}{Ka}\right)\left(\frac{h}{c}\right)^3}{(1-m)(1-\nu^2)} - \frac{(1-m)^2(1+4m+m^2)}{4m^2}\right]$$

$$\bar{A}_2 = \frac{1}{\Delta}\left[\frac{m^4(1+\nu-\log m)}{8(1-m)^2(1-\nu^2)^2}\left(\frac{E}{Ka}\right)^2\left(\frac{h}{c}\right)^6 + \frac{(1-m)^2(1+4m+m^2)}{32} + \frac{m^3\left(\frac{E}{Ka}\right)\left(\frac{h}{c}\right)^3}{64(1-m)(1-\nu^2)} \times \left\{\frac{48}{m^2} - \frac{32}{m} + (1+m)(7+3\nu-\log m^3)\right\}\right] \qquad (5)$$

$$\Delta = 2(1-m)^2(1+4m+m^2) + \frac{3m^3}{(1-\nu^2)} \times (1+\nu-\log m)\left(\frac{1+m}{1-m}\right)\left(\frac{E}{Ka}\right)\left(\frac{h}{c}\right)^3$$

The stress resultants for region $m \le \rho \le 1$ computed from Eq. (4) in accordance with Eqs. (1) will be in error since Eq. (4) is a fair approximation for deflection only. It may be noted that, for $(1 - m) > 0$ as $K \to \infty$, the above expressions reduce to those appropriate for a plate with clamped edge. Similarly, as $m \to 1$ and $K \to \infty$, while $K(1 - m)$ remains finite, we obtain from Eqs. (5) the solution corresponding to the simply supported edge.

Exact Solution

For the region $m \le \rho \le 1$, the load intensity on the bottom and top surfaces is given by $q^+ = Kw + p_o$ and $q^- = p_o - Kw$ where p_o denotes initial uniform compressive stress in the support during assembly, and Eqs. (1) by appropriate elimination result in the following fourth-order equation for w:

$$w^{iv} + (2/\rho)w''' - (w''/\rho^2) + (w'/\rho^3) + \gamma^4 w = 0$$
$$\gamma^4 = 2Ka^4/D \qquad (6)$$

This is the well-known equation for a plate on an elastic foundation, whose solution in terms of $\bar{w}$ is given by

$$\bar{w} = \bar{A}_3\,ber(\gamma\rho) + \bar{A}_4\,bei(\gamma\rho) + \bar{A}_5 ker(\gamma\rho) + \bar{A}_6\,kei(\gamma\rho) \qquad (7)$$

We record the first and second derivatives of $\bar{w}$ as well as the expressions for stress resultants which will be useful for determining the constants of integration;

$$\bar{w}' = \bar{A}_3\gamma ber'(\gamma\rho) + \bar{A}_4\gamma bei'(\gamma\rho) + \bar{A}_5\gamma ker'(\gamma\rho) + \bar{A}_6\gamma kei'(\gamma\rho)$$

$$\bar{w}'' = \gamma^2\left[-\bar{A}_3\left\{bei(\gamma\rho) + \frac{ber'(\gamma\rho)}{\gamma\rho}\right\} + \bar{A}_4\left\{ber(\gamma\rho) - \frac{bei'(\gamma\rho)}{\gamma\rho}\right\} - \bar{A}_5\left\{kei(\gamma\rho) + \frac{ker'(\gamma\rho)}{\gamma\rho}\right\} + \bar{A}_6\left\{ker(\gamma\rho) - \frac{kei'(\gamma\rho)}{\gamma\rho}\right\}\right]$$

$$\bar{V} = m^4\gamma^3[\bar{A}_3bei'(\gamma\rho) - \bar{A}_4ber'(\gamma\rho) + \bar{A}_5kei'(\gamma\rho) - \bar{A}_6ker'(\gamma\rho)] \qquad (8)$$

$$\bar{M}_r = m^2\gamma^2\left[\bar{A}_3\left\{bei(\gamma\rho) + (1-\nu)\frac{ber'(\gamma\rho)}{\gamma\rho}\right\} - \bar{A}_4\left\{ber(\gamma\rho) - (1-\nu)\frac{bei'(\gamma\rho)}{\gamma\rho}\right\} + \bar{A}_5\left\{kei(\gamma\rho) + (1-\nu)\frac{ker'(\gamma\rho)}{\gamma\rho}\right\} - \bar{A}_6\left\{ker(\gamma\rho) - (1-\nu)\frac{kei'(\gamma\rho)}{\gamma\rho}\right\}\right]$$

Identifying the solution in the region $0 \le \rho \le m$ given by Eqs. (2) by subscript 1, and that for the region $m \le \rho \le 1$ given by Eqs. (7) and (8) by subscript 2, the constants of integration $\bar{A}_1$ through $\bar{A}_6$ are determined by satisfying the

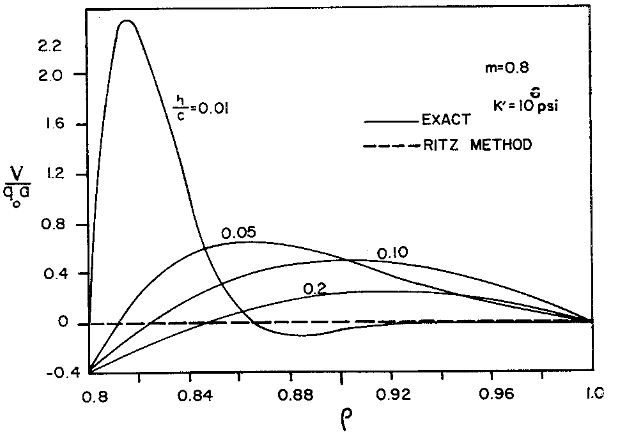

Fig. 2 Distribution of shear stress resultant in the support region.

following boundary and continuity conditions:

$$\left.\begin{aligned} \bar{w}_1 &= \bar{w}_2 \\ \bar{w}_1' &= \bar{w}_2' \\ \bar{w}_1'' &= \bar{w}_2'' \\ \bar{V}_1 &= \bar{V}_2 \end{aligned}\right\} \text{at } \rho = m \qquad (9)$$

$$\bar{V}_2 = \bar{M}_{r2} = 0 \text{ at } \rho = 1$$

The solution thus obtained is exact to within the framework of classical plate theory. The solution of six simultaneous linear algebraic equations arising from the satisfaction of boundary and continuity conditions (9) is greatly facilitated by use of a computer. Several cases were examined by varying the support stiffness and support width. The results are shown in Figs. 2–5.

Results and Discussion

The shear stress resultant varies linearly with ρ in the inner region. Its variation in the support region is shown in Fig. 2 for $K' = 10^6$ psi, where $K' = K(a - c)$, and a support width of 20% of the plate radius. This is the most stringent case to examine when comparing the two solutions. It can be seen that the energy method predicts almost zero value for this resultant in this region, while the more exact solution shows rather large variations of shear stress resultant. Values as large as five times the maximum value of this resultant in the inner region occur according to an exact solution and may go undetected if the energy approach is used. However, it must be pointed out that failure seldom occurs due to shear stress. If, on the other hand, the shear stress resultant is of primary interest in such problems, then the exact solution is indispensable. It should also be emphasized that for $m > 0.8$ (which is the practical range for most supports) and $K' < 10^6$ psi, the aforementioned variation of shear stress resultant will not be as severe. Figure 2 also shows that the shear stress resultant undergoes a change in sign and that the variation is more pronounced for thin plates. This is to be expected, since a thin plate will deflect more and cause a larger reaction from the elastic support.

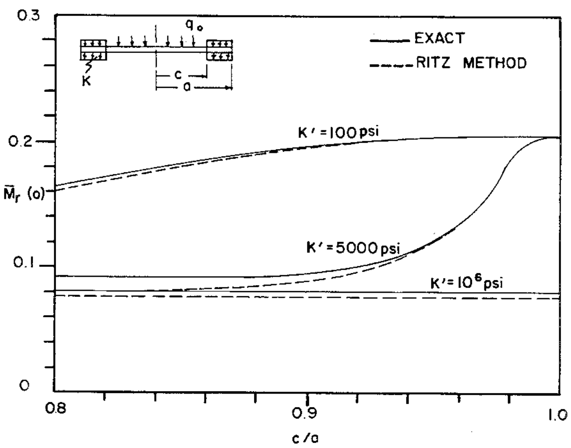

Fig. 3 Variation of central radial moment with support width, $h/c = 0.01$.

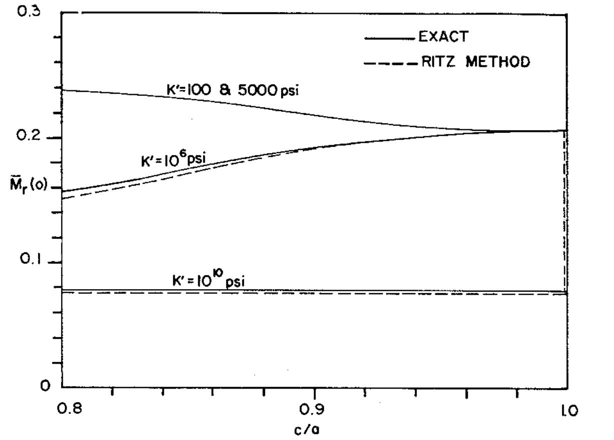

Fig. 4 Variation of central radial moment with support width, $h/c = 0.2$.

Figures 3 and 4 show the variation of central radial moment with the support width for K' values of 100, 5000, and 10^6 psi, and h/c values of 0.01 and 0.2. The agreement between the two solutions is rather remarkable. As the support width approaches zero ($c/a \to 1$), values corresponding to simply supported edge are obtained for all values of K'. It should be noted that as the plate gets thicker, rather large values of K' are required to simulate fixity of the edge. Since the approximate solution given by energy formulation is based on assumptions regarding the deflection in the support region, it might be argued that the effect of these assumptions is negligible on the central radial moment, thus resulting in such close agreement. It seems reasonable, then, to compare the radial moment at the inner periphery of the support. This is done in Fig. 5 for $h/c = 0.01$ and, again, the agreement is seen to be remarkable. For $h/c = 0.2$, even better agreement was found. Finally, it should be mentioned that the deflections given by the two solutions were also compared and found to be in excellent agreement in the entire region of the plate.

Conclusion

The assumed form of plate deflection in the support region, Eq. (4), has been critically examined by comparing the Ritz

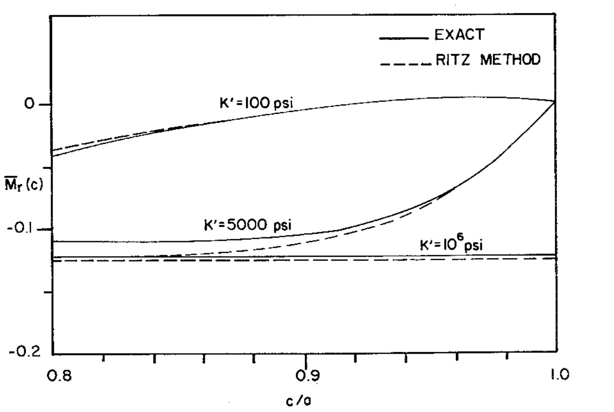

Fig. 5 Variation of edge radial moment with support width, $h/c = 0.01$.

solution with the exact solution (within the framework of classical plate theory). It may be said in conclusion that the underlying assumption leading to Eq. (4) is adequate for support widths as large as 20% of the plate radius, insofar as the deflection and bending moments are concerned. This is borne out by Figs. 3–5. The approximate Ritz solution with its simple form may, therefore, be used with confidence. However, in those problems where the shear stress resultant in the support region is of interest the use of exact solution is imperative.

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Transient Response of a Cylindrical Shell Containing an Orthotropic Core

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A long, circular, cylindrical shell containing an annular, orthotropic, elastic core is subjected to an axisymmetric pressure pulse. The analysis considers the propagation and reflection of stress waves in the core, and the inner core boundary is taken as a rigid reflector or a free surface. Exact formulas in terms of elementary functions for the shell response are obtained for core materials which have specified relationships between the elastic constants. Graphical results from these special cases form closely spaced families of curves and results for an arbitrary orthotropic core can easily be obtained by interpolation.

Nomenclature

a = shell radius
b = inner radius of the core
c = radial dilatational wave speed in the core
c_{ij} = elastic constants of the core; see Eqs. (2a,b)
E = Young's modulus of the shell
E_r,E_θ,E_z = Young's moduli of the core
h = shell thickness
$H(T)$ = Heaviside unit function
I_α,K_α = modified Bessel functions of the first and second kind of order α
P = magnitude of step pressure pulse
q = radial shell displacement
r = radial coordinate
s = Laplace transform variable
t = time
T_D = duration of rectangular pressure pulse
u = radial displacement in the core
γ = shell density
ν = Poisson's ratio of the shell
$\nu_{\theta r},\nu_{z\theta},\nu_{zr}$ = Poisson's ratios of the core
ρ = core density
σ_r,σ_θ = radial and circumferential stress in the core

Dimensionless parameters

c^2 = c_{11}/ρ
k = b/a
T = ct/a
W = $q\omega^2/P$
α^2 = c_{22}/c_{11}
β^2 = $\gamma c^2(1 - \nu^2)/E$
η = r/a
ω^2 = $Eh/(1 - \nu^2)a^2$

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Introduction

COMPOSITE materials have recently improved the efficiency of structural elements for aerospace vehicles. Since many types of composite materials are possible, it is imperative to provide analytical methods which can guide the selection of efficient materials. The present analysis provides criteria for the selection or design of a core material which efficiently reduces the shell strains of a shell-core structure loaded by an axisymmetric pressure pulse. The core is specified as an orthotropic, elastic material. For this axisymmetric problem, the core has six independent material properties, three Young's moduli, and three Poisson's ratios; whereas an isotropic elastic core has only two independent material properties.

Some recent investigations associated with the propagation and reflection of elastic waves in transversely isotropic media have appeared in the literature and are closely related to the present analysis. Eason[1] considered the problem of stress waves emanating from spherical and cylindrical cavities in unbounded transversely isotropic media. Closed-form solutions were obtained for particular solids which had certain relations between the elastic constants. Bickford and Warren[2] extended the work of Ref. 1 and considered the propagation and reflection of elastic waves in isotropic